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Discrete physical systems

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Abstract. Discrete physical (G, Φ, U) systems are defined in terms of a graph G, a set of localised states Φ and a potential U. The dimensionality of the system is defined and shown to be a global property of a graph G. The existence of the thermodynamic limit is investigated and shown to exist if G is dimensional and the potential is short-ranged.

1. Introduction

This paper is concerned with the study of certain aspects of the statistical mechanics of discrete physical systems. A discrete physical system will be taken to consist of three elements: a graph (that is, a set of 'points' with specified 'connections' between the points) which is analogous to the space in which the physics is embedded; a set of localised states associated with each vertex of the graph; and a potential related to the edges of the graph. These elements will be defined more precisely below. We emphasise that the concept of Euclidean space plays no part in the systems we shall study. Our major interest is in those aspects of the system which are affected by the graphical definition of the underlying space. In particular we shall define the concept of dimensionality and show the relation to the thermodynamic limit.

The motivation for this study comes from three sources. First, the use of a graph as the underlying space allows one to construct a mathematical framework for dealing with random arrangements. The need for such a framework has been felt in studies of liquids (Finney 1977). Second, discrete (lattice) models have been used with advantage in the study of hadron physics (Kadanoff 1977, Kogut 1979). Third, the exact enumeration method used in the study of the critical behaviour of the percolation problem (Essam 1973), magnetic systems (Domb 1973) and polymers (McKenzie 1976) makes successful predictions of the behaviour while treating that underlying space as a graph. With regard to methodology and notation we have been influenced by the works of Biggs (1977) and Preston (1974). The physical implications of the ideas presented here will be explored in subsequent publications.

2. Definitions

A graph G of order n is a set of vertices $V(G)=\{x_1, \ldots, x_n\}$, together with a set of edges $E(G)=\{e_1, \ldots, e_m\}$. Each edge can be written as an unordered pair of vertices, thus $e_i=[x_{i1}, x_{i2}]$ where $x_{i1}, x_{i2} \in V(G)$. We shall not be concerned with digraphs,

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multigraphs or graphs with loops. The edge $e = [x, y]$ is incident at x and y. The local degree $\rho(x)$ of x is the number of edges incident at x. If for all $x \in V(G), \rho(x) \leq k_G < \infty$, we say that G is locally bounded. The cardinality of a set A will be denoted by $|A|$.

A path $\mu(x, y)$ is a sequence of edges:

$$\mu(x, y) = ([x, x_1], [x_1, x_2], \ldots, [x_{m-1}, y]).$$

The length of a path, $|\mu(x, y)|$, is the number of edges in the sequence. Let $M_G(\mu(x, y))$ be the set of all paths in G between x and y. The distance $d_G(x, y)$ in G between x and y is defined as $d_G(x, y) = \inf\{|\mathcal{M}|: \mathcal{M} \in M_G(\mu(x, y))\}$. A graph G is connected if $M_G(\mu(x, y)) \neq \phi$ for all $x, y \in V(G)$. If we define $d_G(x, x) = 0$, the set $D_G = \{d_G(x, y): x, y \in V(G)\}$ is a metric for the graph, and in particular the elements of D_G satisfy the triangle inequality

$$d_G(x, y) + d_G(y, z) \geq d_G(x, z).$$

We shall be concerned hereafter with locally bounded, connected graphs whose vertex set is countable.

To introduce a physics on to the graph G we define a set of localised states λ_x associated with each vertex $x \in V(G)$. The set of localised states may be finite, countably infinite, a subset of the real numbers or a Hilbert space or whatever. In this study it will not be necessary to specify λ in detail. Many of the individual features of different physical models arise by specifying λ and associating an algebraic structure with λ, and with the interaction potential to be defined shortly.

A graph B is a subgraph of G, $B \subseteq G$, if $V(B) \subseteq V(G)$ and if $x, y \in V(B)$ with $[x, y] \in E(G)$ then $[x, y] \in E(B)$. Let us denote the indexed class of localised states of G by $\Psi_G = \{\lambda_x: x \in V(G)\}$ and the Cartesian product of Ψ_G by $\Phi_G = \Pi_{x \in V(G)} \lambda_x$. Φ_G will be known as the state space of G. If $V(G) = \{x_i: 1 \leq i \leq n = |V(G)|\}$ then $\omega \in \Phi_G$ can be represented by the n-tuple $\omega = (\nu_1, \nu_2, \ldots, \nu_n)$ with $\nu_i \in \lambda_i$. If $B \subset G$, so that $V(B) = \{x_1, \ldots, x_p\}$, then

$$\pi_B(\omega \in \Phi_G) = (\nu_{x_1}, \nu_{x_2}, \ldots, \nu_{x_p}) \in \Phi_B,$$

that is π projects the state of G into that of B. We shall often use the notation ω_G meaning $\omega \in \Phi_G$.

We suppose now that there exists a potential function U which maps Φ_G into the real numbers, $U: \Phi_G \to \mathbb{R}$. A discrete physical system (G, Φ_G, U) consists therefore of the graph G, the state space Φ_G and the potential U. To each $\omega \in \Phi_G$ we associate a positive weight $I(\omega) = \exp(U(\omega))$. We then define the probability of a Gibbs state of the system (G, Φ_G, U) by

$$p_G(\omega) = I(\omega) \Big/ \sum_{\omega \in \Phi_G} I(\omega).$$

The partition function of the system is determined by the normalising constant namely

$$Z_G = \sum_{\omega \in \Phi_G} I(\omega).$$

The equilibrium properties of the system follow in the usual way from the partition function and its log derivatives.

Following Preston (1974) we define an interaction potential $J_A: \Phi_A \to \mathbb{R}$ by

$$J_A(\omega) = \sum_{B \subseteq A} (-1)^{|V(A) - V(B)|} U_B(\pi_B(\omega_A))$$

whence, using Mobius inversion or the principle of inclusion–exclusion (Rota 1964, Feller 1960)

$$U_G(\omega) = \sum_{A \subset G} J_A(\pi_A(\omega_G)).$$

Here $U_G(\omega)$ is the 'global' weight attached to state ω while the $J_A(\omega)$ conveniently divide $U_G(\omega)$ into a sum of weights attached to each vertex, edge, and so on of the graph G.

For physical reasons there is a more-or-less intimate connection between the interaction potentials J and the graph G. One may view the vertices of the graph as physical entities such as atoms or spins. The edges of the graph then indicate the atoms which interact directly, or vice versa, the nearest-neighbour potential determines the edges of the graph. Triplet forces, quadruplet forces, and so on may be included if desired, but the graph model becomes unsuitable for practical purposes if one continues too far on those lines.

3. Graph dimensionality

Certain properties of phase transitions, notably critical exponents, appear to depend solely on the dimensionality of the underlying space, or lattice, for a given model. The dimensionality of lattice models is defined with reference to the embeddibility of the lattice in Euclidean d-dimensional space, or at least into $\mathbb{Z}^d$ where $\mathbb{Z}$ is the set of integers. We require here a definition of the dimensionality of a graph independent of a relation to Euclidean space, that is a definition in terms of the graph alone. We have chosen to base the idea of dimensionality on how fast the graph grows from a given vertex. We shall show that the growth rate for infinite graphs is independent of the choice of origin, so that dimensionality is a property of the graph as a whole, not a local property.

Let G be a locally bounded, infinite, connected graph with maximum local degree k_G. Choose $\alpha \in V(G)$ and let us define the sets $X_i^\alpha \subset V(G)$ by $X_i^\alpha = \{x : x \in V(G), d_G(x, \alpha) = i, i \geq 0\}$. We shall call X_n^α the nth ring with respect to α and the sequence $(X_o^\alpha, X_1^\alpha, \dots)$ an α-sequence. Since G is connected no X_n^α is empty and the α-sequence does not terminate. We define also the sets $Y_n^\alpha = \bigcup_{i=0}^n X_i^\alpha$ known as the nth ball with respect to α. It is convenient to use the notation $c_n^\alpha = |X_n^\alpha|$ and $C_n^\alpha = |Y_n^\alpha|$, that is $C_n^\alpha = \Sigma_{i=0}^n c_i^\alpha$. Since G is connected, $c_n^\alpha \geq 1$ and $C_n^\alpha \geq n$.

We shall now say that G is *regular* if for all $\alpha \in V(G)$ the growth rate, c_n^α, can be described by a real valued strictly positive, monotonic function of N, $\phi_\alpha(n)$; that is, for $n \geq 1$ there exist finite non-zero real numbers M and N such that

$$N_\alpha \leq \frac{c_n^\alpha}{\phi_\alpha(n)} \leq M_\alpha. \tag{1}$$

It is convenient to introduce an abbreviated notation for (1). Thus given two sequences $F = (f_n : n \in \mathbb{Z}, f_n > 0)$ and $H = (h_n : n \in \mathbb{Z}, h_n > 0)$ we shall say that

$$f_n \sim h_n, [N, M; n_0],$$

if for all $n \geq n_0$

$$0 < N \leq f_n/h_n \leq M < \infty.$$

The relation $\sim$, or 'behaves as', is an equivalence relation since if $f_n \sim h_n$, $[N, M; n_0]$, then $h_n \sim f_n$, $[1/M, 1/N; n_0]$; and if $f_n \sim h_n$, $[N_0, M_0; n_0]$ and $h_n \sim l_n$, $[N_1, M_1; n_1]$, then $f_n \sim l_n$, $[\inf(N_0, N_1), \sup(M_0, M_1); \sup(n_0, n_1)]$. We note that if two sequences f_n and h_n are such that $f_n \not\sim h_n$ then either $f_n/h_n \to \infty$ or $h_n/f_n \to \infty$ as $n \to \infty$.

It is clear that if $c_n^\alpha \sim \phi_\alpha(n)$, $[N_\alpha, M_\alpha; 1]$, then there exists a function $\psi_\alpha(n) = \sum_{i=1}^n \phi_\alpha(i)$ such that $C_n^\alpha \sim \psi_\alpha(n)$, $[N_\alpha, M_\alpha; 1]$.

Our main result of this section is that if G is regular then the functions $\phi_\alpha(n)$ do not depend significantly on the choice of origin α. We shall require the following intermediate result.

Lemma 1. For all $n \geq n_0$, $\phi_\alpha(n) \not\sim \phi_\beta(n)$ if and only if $\psi_\alpha(n) \not\sim \psi_\beta(n)$.

Proof.

(i) Suppose $\phi_\alpha(n) \not\sim \phi_\beta(n)$. Then for all $n \geq n_0$ there is a $K(n_0)$ which can be as large as one chooses such that $\phi_\alpha(n) > K\phi_\beta(n)$. Then

$$\begin{aligned}
\psi_\alpha(n) &= \sum_{i=1}^{n} \phi_\alpha(i) = \sum_{i=1}^{n_0-1} \phi_\alpha(i) + \sum_{i=n_0}^{n} \phi_\alpha(i) \\
&> K \sum_{i=n_0}^{n} \phi_\beta(i) \qquad \text{since } \phi_\alpha(i) \geq 0 \\
&= K\Big(\psi_\beta(n) - \sum_{i=1}^{n_0-1} \phi_\beta(i)\Big) \\
&= K\psi_\beta(n)(1 - g_\beta(n_0)/\psi_\beta(n)) \\
&> K'\psi_\beta(n),
\end{aligned}$$

where $K > K'$ since $g_\beta(n_0)$ is a constant and $\psi_\beta(n) \geq (n-1)$.

(ii) Suppose $\psi_\alpha(n) > K\psi_\beta(n)$; then

$$\sum_{i=1}^{n} (\phi_\alpha(i) - K\phi_\beta(i)) > 0.$$

But at least one term in the sum on the right-hand side of the above equation must be positive, whence

$$\phi_\alpha(n_1) > K\phi_\beta(n_1)$$

for some n_1 with $1 \leq n_1 \leq n$. But since $\phi(n)$ is monotonic we must have

$$\phi_\alpha(n) > K\phi_\beta(n) \quad \text{for all } n \geq n_1.$$

We shall now prove

Proposition 1. If G is a connected, locally bounded, regular infinite graph and

(i) $\alpha, \beta \in V(G)$ with $d_G(\alpha, \beta) = m < \infty$

(ii) $c_n^\alpha \sim \phi_\alpha(n)$, $[N_\alpha, M_\alpha; 1]$

(iii) $c_n^\beta \sim \phi_\beta(n)$, $[N_\beta, M_\beta; 1]$

then $\phi_\alpha(n) \sim \phi_\beta(n)$, $[g_0, g_1; n_0]$.

Proof. We shall show that the assumption that $\phi_\beta(n)/\phi_\alpha(n) > K$ where K can be as large as one likes, that is $\phi_\alpha(n) \not\sim \phi_\beta(n)$, leads to a contradiction and hence

$\phi_\beta(n)/\phi_\alpha(n) \leqslant g_1 < \infty$. But from symmetry we must also have $\phi_\alpha(n)/\phi_\beta(n) \leqslant 1/g_0 < \infty$, and hence

$$0 < g_0 \leqslant \phi_\beta(n)/\phi_\alpha(n) \leqslant g_1 < \infty,$$

that is $\phi_\alpha(n) \sim \phi_\beta(n)$.

We note first, as a consequence of lemma 1, that if $\phi_\beta(n)/\phi_\alpha(n) > K$ for all $n \geqslant n_0$ where K can be made as large as one chooses, then there is an n_1 and a $K' > K$ such that $\psi_\beta(n)/\psi_\alpha(n) > K'$ for all $n \geqslant n_1$.

To prove the contradiction we shall show that $\psi_\beta(n)/\psi_\alpha(n)$ is bounded above and hence $\phi_\beta(n)/\phi_\alpha(n) \not> K$. Thus suppose $C_n^\alpha \sim \psi_\alpha(n)$, $[N_\alpha, M_\alpha; 1]$ and $C_n^\beta \sim \psi_\beta(n)$, $[N_\beta, M_\beta; 1]$. Define the set $P_n^{\alpha\beta} = Y_n^\alpha \cap Y_n^\beta$ and choose $n_{\alpha\beta} > m$ so that $P_n^{\alpha\beta}$ is not empty (figure 1). Let $C_n^P = P_n^{\alpha\beta} - P_{n-1}^{\alpha\beta}$ so that since $P_{n-1}^{\alpha\beta} \subset P_n^{\alpha\beta}$ we have

$$|C_n^P|/|P_n^{\alpha\beta}| \leqslant 1.$$

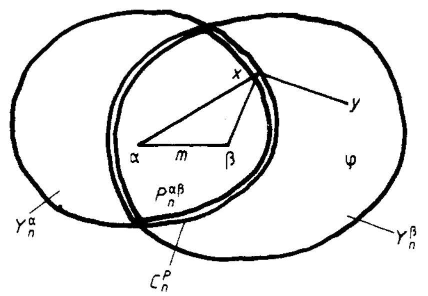


Figure 1. Venn diagram of the the sets introduced in the proof of Proposition 1.

Furthermore, since $P_n^{\alpha\beta} \subset Y_n^\alpha$ we must have

$$|P_n^{\alpha\beta}| \leqslant C_n^\alpha \leqslant M_\alpha \psi_\alpha(n). \tag{2}$$

We shall now show that

$$N_\beta \psi_\beta(n) \leqslant C_n^\beta \leqslant |C_n^P| k_G^m + |P_n^{\alpha\beta}|.$$

Consider the set $Q = Y_n^\beta - P_n^{\alpha\beta}$. If $y \in Q$ then any path $\mu(\beta, y)$ must contain a member of C_n^P. Let x be such a vertex and $\mu(\beta, \ldots, x, \ldots, y)$ be one of the shortest paths between β and y, so that $d_G(\beta, y) = d_G(\beta, x) + d_G(x, y) \leqslant n$. From the definition of C_n^P either $d_G(x, \alpha) = n$ or $d_G(x, \beta) = n$ of which the second possibility must be excluded. But from the triangle inequality we have

$$d_G(x, \beta) + d_G(\alpha, \beta) \geqslant d_G(x, \alpha) = n$$

whence

$$\begin{aligned} d_G(x, y) = d_G(\beta, y) - d_G(\beta, x) &\leqslant n - (d_G(\alpha, x) - d_G(\alpha, \beta)) \\ &= n - (n - m) \\ &= m. \end{aligned}$$

Thus any vertex in Q can be reached from a vertex in C_n^P in, at most, m steps. But since the maximum local degree of G is k_G, the number of vertices in Q can be most $|C_n^P| k_G^m$.

Hence we obtain

$$C_n^\beta \leqslant |C_n^P| k_G^m + |P_n^{\alpha\beta}|$$

and (3)

$$N_\beta \psi_\beta(n) \leqslant |C_n^P| k_G^m + |P_n^{\alpha\beta}|.$$

Putting the inequalities (2) and (3) together we obtain

$$\frac{\psi_\beta(n)}{\psi_\alpha(n)} \leqslant \frac{M_\alpha}{N_\beta} \frac{|C_n^P| k_G^m + |P_n^{\alpha\beta}|}{|P_n^{\alpha\beta}|}$$

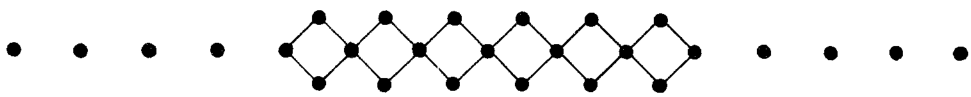
$$< \frac{M_\alpha}{N_\beta} ((k_G^m + 1).$$

Hence for any finite value of m, $\psi_\beta(n)/\psi_\alpha(n)$ is bounded above (which proves the contradiction).

Corollary. The growth rate $\phi_\alpha(n)$ is characteristic of the graph G, since any vertex of G can be reached from α in a succession of finite steps. Hence we can write $\phi_G(n) \sim \phi_\alpha(n)$.

If $\phi_G(n) \sim n^{d-1}$ we say that the graph is a d-dimensional. In this case, the number of vertices in the ball is given by $\psi_G(n) \sim n^d$. Thus $\phi_G(n)/\psi_G(n) \sim 1/n$ which tends to zero as n tends to infinity and hence the ratio of 'surface' to 'volume' tends to zero with increasing size.

A graph G is *homogeneous* (or vertex transitive) if every vertex is equivalent to every other vertex (in the sense that one obtains the same canonical description of the graph (McKenzie 1975) whatever vertex one chooses to label as vertex 1). We shall term a homogeneous locally bounded infinite graph a *lattice.* Clearly Proposition 1 is true trivially for all lattices. We note that our definition of dimensionality agrees with the accepted notion for the common lattices (table 1). However, we are in a position to cope with graphs such as



for which $c_n \sim 1$, [2, 4; 3], and hence is one-dimensional. Furthermore, for any connected graph $c_n \geqslant 1$ so that the lowest dimensionality is one. It is not inconceivable that there exist graphs with non-integral dimensionality.

If $\phi_G(n) \sim \mu^n$ with $\mu > 1$, we say that G is a Bethe graph. In this case $\psi_G(n) = \Sigma_i \phi_G(i) \sim \mu^n$ so that $\phi_G(n)/\psi_G(n) \sim A(\mu) =$ a constant. That is for Bethe graphs the 'surface' is as large as the 'volume'. We note that for locally bounded graphs with maximum local degree k_G, the number of vertices in ring X_n is given by

$$c_n \leqslant (k_G - 1)c_{n-1} \leqslant (k_G - 1)^n$$

so that the fastest growth rate is μ^n with $\mu = k_G - 1$.

The concept of dimensionality or growth rate is independent of how well the graph is connected. For example, the growth rate of G can be defined by choosing a spanning tree rooted at a particular vertex as origin. The remaining edges of G can be replaced, remembering that the maximum local degree is k_G, to construct a wide variety of

Table 1. Dimensionality of common lattices.

Lattice G	$\phi_G(n)$	Dimensionality
Linear chain	2	1
Honeycomb	$3n$	2
Single quadratic	$4n$	2
Triangular	$6n$	2
Diamond	$\frac{1}{4}(10n^2+7+(-1)^n)$	3
Single cubic	$4n^2+2$	3
Body-centred cubic	$6n+2$	3
Face-centred cubic	$10n^2+2$	3
4-d hypercubic	$\frac{8}{3}(n^2+2)$	4
5-d hypercubic	$\frac{2}{3}(2n^4+10n^2+3)$	5

different graphs with varying degrees of connectivity. We shall return to the question of the connectivity of an infinite graph, which is related to existence of a phase transition for the graph, in a subsequent paper.

Our definition of dimensionality is closely related to the problem of whether a random walk on a graph is recurrent. A generalisation of this problem has been studied by Nash-Williams (1959) who showed that, in our notation, the graph is recurrent if $1/\phi_G(n)$ diverges. Thus, with $\phi_G(n) \sim n^{d-1}$, all graphs with dimension less than or equal to two are recurrent. Nash-Williams used a generalisation of the concept of rings, by defining a sequence of finite non-empty nested subgraphs of the graph rather than the α-sequence employed here. We prefer to use the α-sequence because it is easier to relate to applications and because the α-sequence is directly related to the canonical description of the graph. Thus to study the effect of changing from one discrete physical system to another by changing the graph alone it is most useful to have a definition of dimensionality which is a property of the description of the graph.

The definition of dimensionality of Biggs (1977) namely the repetition of a basic figure, that is a finite graph, under the action of a group is related to the definition given here in the sense that a given graph will have the same dimensionality according to both definitions. For example, one can deduce that for the d-dimensional hypercubic lattices, the appropriate group implies that the generating function

$$\sum_n c_n x^n = \left(\frac{1+x}{1-x}\right)^d,$$

with cyclic boundary conditions. This generating function gives the appropriate coefficients quoted in table 1. However, the definition of dimensionality in terms of a group is only applicable to graphs which have particular symmetry properties whereas the definition in terms of growth rate is not so restricted.

4. The thermodynamic limit

A discrete physical system (G, Φ, U) is related to the world of experimental physics by defining the free energy by

$$F_G = \ln Z_G. \tag{4}$$

To be sensible, physically, the free energy should be an extensive quantity, that is F should be proportional to the size of the system. Here the size of the system is the number of vertices in the graph. The physical volume of the system is defined by associating a volume v with each vertex, or possibly a different volume with each localised state. Hence for a finite graph we should obtain $F_G \propto |V(G)|$.

For an infinite graph we must specify how the graph becomes infinite. We propose to use the definition of an α-sequence. Let us define therefore the sequence of nested sub-graphs $G_n^\alpha \subset G$ by $V(G_n^\alpha) = Y_n^\alpha$, $E(G_n^\alpha) = \{[x, y]: x, y \in Y_n^\alpha, [x, y] \in E(G)\}$ so that one obtains G in the limit as $n \to \infty$ of G_n^α. We wish to investigate under what circumstances

$$f_G = \lim_{n\to\infty} (1/|G_n^\alpha|) \ln Z_{G_n^\alpha} \tag{5}$$

exists. This approach to the thermodynamic limit will be known as taking the limit in the ring sense. It turns out that the major requirement for the existence of the limit is that the interaction potentials J_B should be short-ranged and of bounded variation. It will also become apparent that the existence of the limit is independent of the choice of α.

We shall say that a potential is short-ranged if there is a $q < \infty$ such that for all $\omega \in \Phi_G$ and for any $x \in V(G)$

$$\Bigg\| \sum_{\substack{A \supset x \\ A \subset G \\ A \neq x}} J_A(\pi_A \omega_G) \Bigg\| < q \tag{6}$$

where $\|z \in \mathbb{R}\|$ is the absolute value of z. Clearly (6) covers a very wide range of possible potentials. However, one requirement on the potential to satisfy (6) is that $J_A(\omega)$ is bounded above and below. Furthermore, with $|V(A)| = n$, J_A gives the contribution of n-body forces. To satisfy (6) the contribution of n-body forces must decrease sufficiently fast as n increases. The easiest way in which this could be accomplished is that $J_A = 0$ for n greater than some specified limit. One should note that the contribution of all subgraphs for a given graph A is the number of embeddings of A on G which include x and that this number is finite since G is locally bounded.

The requirement (6) is a condition on the interaction potential not including the contribution from each vertex singly. We shall also require the condition that

$$\sum_{\omega \in \lambda_x} \exp(J_x(\omega)) = g_x \tag{7}$$

exists, where $x \in V(G)$ and λ_x is the set of localised states. Here g_x is merely the partition function of a single vertex.

We are now in a position to state

Proposition 2. If (G_n^α, Φ, U) form a sequence of discrete physical systems in the ring sense with $n \geq 0$, and U is defined by an interaction potential J_A, $A \subset G_n^\alpha$, which satisfies conditions (6) and (7) above, then $f_n = \ln Z_{G_n}/C_n^\alpha$ is bounded above and below, and if $G = \lim_{n\to\infty} G_n^\alpha$ is d-dimensional then $\lim_{n\to\infty} f_n$ exists.

Proof. Consider

$$Z_{G_n^\alpha} = \sum_{\omega \in \Phi_{G_n^\alpha}} I(\omega) = \sum_{\omega_1 \in \Phi_{G_{n-1}^\alpha}} \sum_{\omega_2 \in \Phi_{X_n^\alpha}} I(\omega_1) \exp\Bigg[\sum_{\substack{A \subset G_n^\alpha \\ V(A) \cap V(X_n^\alpha) \neq \phi}} J_A(\pi_A(\omega_1 \times \omega_2)) \Bigg]$$

$$Z_{G_n^\alpha}=\sum_{\omega_1} I(\omega_1)\sum_{\omega_2}\exp\left[\sum_{x\in X_n^\alpha} J_x(\pi_x(\omega_2))+\sum_{A\neq x} J_A(\pi_A(\omega_1\times\omega_2))\right]$$

$$=\sum_{\omega_1} I(\omega_1)\sum_{\omega_2}\exp\left[\sum_{x\in X_n^\alpha} J_x(\pi_x(\omega_2))+\sum_{x\in X_n^\alpha}\sum_{A\supset x}\frac{J_A(\pi_A(\omega_1\times\omega_2))}{|V(A)\cap V(X_n^\alpha)|}\right].$$

Since $|V(A)\cap V(X_n^\alpha)|\geqslant 1>0$, we must have, using (6),

$$\sum_{\omega_1} I(\omega_1)\sum_{\omega_2}\exp\left[\sum_{x\in X_n^\alpha} J_x(\pi_x(\omega_2))-\sum_{x\in X_n^\alpha} q\right]\leqslant Z_{G_n^\alpha}$$

$$<\sum_{\omega_1} I(\omega_1)\sum_{\omega_2}\exp\left[\sum_{x\in X_n^\alpha} J_x(\pi_x(\omega_2))+\sum_{x\in X_n^\alpha} q\right]$$

which becomes

$$\sum_{\omega_1} I(\omega_1)\prod_{x\in X_n^\alpha}\sum_{\omega\in\lambda_x}\exp[-q+J_x(\omega)]<Z_{G_n}<\sum_{\omega_1} I(\omega_1)\prod_{x\in X_n^\alpha}\sum_{\omega\in\lambda_x}\exp[q+J_x(\omega)].$$

Using (7) we obtain

$$\sum_{\omega_1} I(\omega_1)\prod_{x\in X_n^\alpha}(\mathrm{e}^{-q}g_x)<Z_{G_n^\alpha}<\sum_{\omega_1} I(\omega_1)\prod_{x\in X_n^\alpha}(\mathrm{e}^{q}g_x)$$

or

$$Z_{G_{n-1}^\alpha}(\mathrm{e}^{-q}g_x)^{c_n^\alpha}<Z_{G_n^\alpha}<Z_{G_{n-1}^\alpha}(\mathrm{e}^{q}g_x)^{c_n^\alpha}. \tag{8}$$

Iterating (8) for $n\geqslant 0$ we obtain

$$Z_{G_0^\alpha}P^{C_n^\alpha}<Z_{G_n^\alpha}<Z_{G_0^\alpha}Q^{C_n^\alpha}$$

where, since G_0^α is the single vertex α, $Z_{G_0^\alpha}=g_x$, and $P=\mathrm{e}^{-q}g_x$ and $Q=\mathrm{e}^{q}g_x$. Hence

$$\frac{\ln g_x}{C_n^\alpha}+\ln P<\frac{\ln Z_{G_n^\alpha}}{C_n^\alpha}<\frac{\ln g_x}{C_n^\alpha}+\ln Q \tag{9}$$

and since C_n^α increases indefinitely with n, (9) shows that $\ln Z_{G_n^\alpha}/C_n^\alpha$ is bounded.

From (8) we also obtain the inequalities

$$c_n^\alpha\ln P<\ln Z_{G_n^\alpha}-\ln Z_{G_{n-1}^\alpha}<c_n^\alpha\ln Q$$

whence

$$\frac{c_n^\alpha}{C_n^\alpha}\left(\ln P-\frac{\ln Z_{G_{n-1}^\alpha}}{C_{n-1}^\alpha}\right)<\frac{\ln Z_{G_n^\alpha}}{C_n^\alpha}-\frac{\ln Z_{G_{n-1}^\alpha}}{C_{n-1}}<\frac{c_n^\alpha}{C_n^\alpha}\left(\ln Q-\frac{\ln Z_{G_{n-1}^\alpha}}{C_{n-1}^\alpha}\right)$$

Using (9) and the fact that $\ln g_x/C_{n-1}^\alpha\to 0$ as $n\to\infty$ we obtain

$$\left\|\frac{\ln Z_{G_n^\alpha}}{C_n^\alpha}-\frac{\ln Z_{G_{n-1}^\alpha}}{C_{n-1}^\alpha}\right\|<\frac{c_n^\alpha}{C_n^\alpha}(\ln Q-\ln P). \tag{10}$$

If G is d-dimensional $c_n^\alpha/C_n^\alpha\sim 1/n$ and the right-hand side of (10) tends to zero as n tends to infinity and the limit f_n exists. For Bethe graphs, the ratio c_n^α/C_n^α tends to a constant so that the thermodynamic limit does not necessarily exist for discrete physical systems based on such graphs.

Corollary. It should be evident from the above proof that the proof is unchanged on replacing α by β. Thus the proposition is true independently of the choice of origin α and hence the thermodynamic limit is a property of the discrete physical system.

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